The University of St. Thomas

New Mathematical Principles Applied to Card Tricks

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Low Down Triple Dealing

Consider this demonstration of mathemagic:

A quarter of a deck of cards is handed to a spectator, who is invited to shuffle freely. She is asked to call out her favourite ice-cream flavour; let's suppose she says, "Chocolate."

Take the cards back, and deal them into a pile, one card for each letter of "chocolate," before dropping the remainder on top.

This spelling routine is repeated twice more (three deals total). Emphasize how random the dealing was, since the cards were shuffled and you had no control over the named ice-cream flavour. Yet, you correctly name the top card in the pile at the conclusion of the triple dealing. *Proof Without Words*

Alpha Omega

1 6

4 3

5 2

is remarkable for several reasons:

- It's 1, 2, 3, 4, 5, 6, laid out in a nice geometric pattern
- Triple sums centered on even numbers are always 10
- Triple sums centered on odd numbers are always 11
- Amazingly, it's 1, 2, 3, 4, 5, 6 in alphabetical order!

From Alpha–Omega to One—Six

This bracelet has yet another magic property:

Hold a packet of face-down cards in one hand, in order
5, 4, 1, 6, 3, 2 from the top.

- Move one card from top to bottom for each letter in the words "count it"—thus bringing the 4 to the top.
- Move one card from top to bottom and set the next card aside (it's the Ace)
- Move two cards from top to bottom and set the next card aside (it's the 2)
- Move three cards from top to bottom and set the next card aside (it's the 3)
- Keep going until only the 6 remains.

The Persistence of Six

It's easy to show that any magic bracelet which alternately forces triple sums of n or $n+1$ must have length 6.

These must basically be of the form:

$$a, \quad b, \quad n-a-b, \quad a+1, \quad b-1, \quad n+1-a-b$$

To get an example with distinct positive beads the smallest possible value for n is 10, since $1+2+3+4+5+6=21$.

The alternating triple sums of any such bracelet are 10 and 11.

Upon reflection (or rotation, or both), every such bracelet is [5 4 1 6 3 2], as seen earlier.

Mantric Six

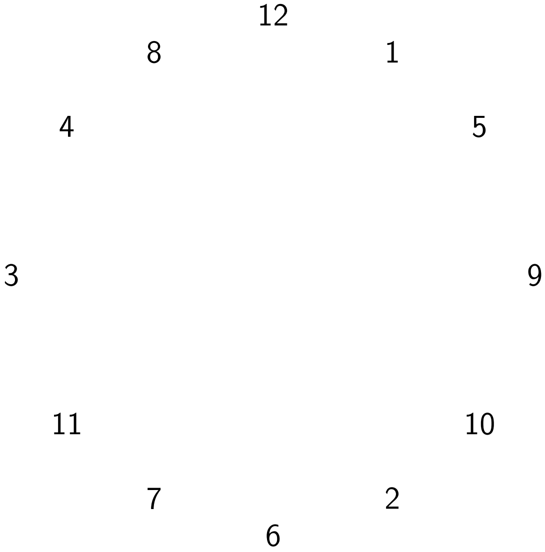
If we drop the distinctness restriction, then there are five other bracelets with triple sums 10 and 11, such as [5 4 2 4 5 1].

Mantric Six: The perfect [1 4 5 2 3 6] may be obtained by starting with 1, and three times repeating the modular mantra

> *"First add 3, and then add 1."*

What Next? Generalize!

A Magic Timepiece Influenced By Martin Gardner (Celebrating His 93rd Birthday Incidentally!)



A Magic Timepiece Influenced By Martin Gardner (Celebrating His 93rd Birthday Incidentally!)

- Quadruple sums of adjacent beads cycle through the values 25, 26, 27.
- This fact permits the performance of a trick as follows: Remove an Ace, 2, 3, ..., 10, Jack and Queen from a blue-backed deck of cards, and deal them face down into a circle, representing the hours of a crazy clock. Have any adjacent four cards selected, and their values summed: it's always one of 25, 26 or 27. Now pick up a red-backed deck: the 26th card from the top is forcible (and may be predicted in advance).
- An appropriate *Card Colm* was published on 21 October.
- It's hard to remember [1 5 9 10 11 2 6 7 11 3 4 8 12] *in order*.

  A catchy mnemonic would really help . . .

The Birthday Card Match Principle

How many cards, picked randomly from a standard deck of 52, are required so that there is a greater than 50% chance of getting at least one pair with the same value?

Let's call this desirable phenomenon a birthday card match—that's *"birthday" card match* and not *"birthday card" match*.

This problem can be tackled using a common approach to the classic Birthday Problem, which concerns the number of people required to ensure a greater than 50% chance of having at least one birthday match.

The surprisingly small answer there is 23 people!

The Birthday Card Match Principle

The key to estimating such probabilities is to turn things around, and focus on the chances of there being no match, noting that

Prob(at least one match) = 1 - Prob(no match).

If k cards are picked at random, then since a deck contains four cards of each value, it's clear that for $2 < k < 14$ we get:

$$1 - \frac{52}{52} \times \frac{48}{51} \times \frac{44}{50} \times \ldots \times \frac{52-4k+4}{52-k+1}.$$

It turns out that we need to pick at least 6 cards to be at least 50% sure of a birthday card match.

Given 8 or 9 cards, there is a high probability (89% or 95%) of a match, and with 10 cards, it's very likely (98%) to occur.

Better Poker Hands with Bill Simon

When you glance at the faces of the cards you are given, *assuming that there is at least one matching pair*, the basic idea is to ensure that by dropping clumps of cards casually, the "winning cards" are among the bottom four.

If you play your cards right, you can ensure that later on, you get those winning cards.

Forget about the first two cards for now, and suppose you have just eight cards.

We need a wonderful observation from Bill Simon's 1964 book *Mathematical Magic*, which included an effect called "The Four Queens."

Bill Simon's Sixty-Four Principle

It is possible to give the illusion of multiple free choices to a spectator, while deciding how to split up a packet of eight cards into two piles of four.

In fact, you retain control of the division in one key sense: the top four cards all end up in the first pile.

You can even have the spectator handle the cards throughout, after you appear to have shuffled them.

Hence, if you start with 4 red cards of top of 4 black ones, the piles maintain that colour separation, with the reds in the first pile.

Bill Simon's Sixty-Four Principle

We now describe how Simon's separation scheme works in practice: an effective way to follow along is to work with a face-up packet of 4 reds followed by 4 blacks.

The spectator is given the choice of putting the top card on the table to start Pile A, or tucking the top card underneath the rest of the packet. The second card then goes wherever the first one did not (under the packet if the first one went to the table, and vice versa).

Overall, one of the first 2 cards starts Pile A, and the other goes to the bottom of the packet.

Give the spectator the exact same free choice for the second pair of cards.

Bill Simon's Sixty-Four Principle

Unsuspected by most is the fact that Pile A now contains 2 red cards, and the retained packet consists of 4 blacks followed by 2 reds.

Next, the spectator is asked to make similar choices to determine 2 cards for Pile B. Of course, the result is that 2 blacks start that pile and the retained packet consists of 2 reds followed by 2 blacks.

(Note that at this stage we have a scaled down version of the original packet.)

Now, the spectator uses the same procedure to pick just 1 card for Pile A, and finally 1 for Pile B, unwittingly maintaining the colour separation.

Bill Simon's Sixty-Four Principle

Pause to recap what has happened, claiming "Six times, I gave you completely independent free choices. That's two to the power of six, or sixty-four, different things that could have happened so far."

The last 2 cards are a red followed by a black, and you must have the first added to Pile A, and the second to Pile B.

This can be done either by casually asking the spectator to deal them that way, or you can come up with some magicians force to achieve the same result (Simon suggested a specific one).

Of course, this can also be applied to packets of size 16 (or 32), if suitable modifications are made.

It's as easy as Aodh, Bea, C & D

A MATHEMAGICAL DRAMA

The cast: mathemagicans Aodh and Bea, an invariant volunteer C, and a shuffled deck of cards D.

Scene 1: All but Bea are on a sparsely furnished stage. C selects some cards from D and gives them to Aodh, before leaving stage left. Aodh glances at the cards, places them in a row on the table, with one card face concealed, and then exits stage left.

Scene 2: Bea, who has witnessed nothing prior to this display, enters stage right, glances at the cards on the table, and after a suitable pause, promptly reveals the identity of the hidden card.

Fitch Cheney's Five-Card Trick

A volunteer from the crowd chooses any five cards at random from a shuffled deck, and hands them to you so that nobody else can see them.

You glance at them briefly, and hand one card back, which the volunteer then places face down on the table to one side.

You quickly place the remaining four cards face up on the table, in a row from left to right.

Your confederate, who has not been privy to any of the proceedings so far, arrives on the scene (e.g., is called in from another room), inspects the faces of the four cards, and promptly names the hidden fifth card.

Fitch Cheney's Five-Card Trick

This superb effect is usually credited to mathematician William Fitch Cheney Jnr (1904-1974), who in 1927 received the first PhD in Math awarded by MIT.

A keen magician all his life, Cheney was also Editor of the Puzzle Section of the *American Mathematical Monthly* from 1930 to 1940.

Note that *you* get to choose which card to hand back, and later on, in what order to place the remaining four cards.

The first condition can be worked around actually ... ask how later if this interests you!

Fitch Cheney's Five-Card Trick

Three main ideas make this magic possible:

1. The pigeonhole principle guarantees that (at least) two of the five cards are of the same suit.

WLOG you have two Clubs.

One Club is handed back, and by placing the remaining four cards in some particular order, you effectively tell your confederate the identity of the Club you just handed back.

2. You can use one designated position (e.g., the first) of the four available on the table for the retained Club—which determines the suit of the hidden card

Use the other three positions for the placement of the remaining cards, which can be arranged in $3! = 6$ ways.

Fitch Cheney's Five-Card Trick

If you and your confederate agree in advance on a one-to-one correspondence between the six possible permutations and $1, 2, \ldots, 6$, then you can communicate one of six things.

What *can* one say about these other three cards? Not much—for instance, some or all of them could be Clubs too, or there could be other suit matches!

However, one thing *is* certain: they are all distinct, so with respect to some total ordering of the entire deck, one of them is LOW, one is MEDIUM, and one is HIGH.

This permits for an unambiguous and easily remembered way to communicate a number between 1 and 6.

Fitch Cheney's Five-Card Trick

But surely 6 isn't enough?

The hidden card could in general be any one of 12 Clubs!

This brings us to the third main idea:

3. You must be careful as to which card you hand back.

Considering the 13 possible card values, 1 (Ace), $2, 3, \ldots, 10$, J, Q, K, as being arranged clockwise on a circle, we see that the two suit match cards are at most 6 values apart, i.e., counting clockwise, one of them lies at most 6 vertices past the other.

Give this "higher" valued Club back to the victim, which they then hide. You'll use the "lower" Club and the other three cards to communicate the identity of this hidden card to your confederate.

Fitch Cheney's Five-Card Trick

For example, if you have the $2\clubsuit$ and $8\clubsuit$, then hand back the $8\clubsuit$, but if you have the $2\clubsuit$ and $J\clubsuit$, hand back the $2\clubsuit$.

In general, you save one card of a particular suit and need to communicate another of the same suit, whose numerical value is k higher than the one you make available, for some integer k between 1 and 6 inclusive.

Put this total linear ordering on the whole deck:

$$A\clubsuit, 2\clubsuit, \ldots, K\clubsuit, A\diamondsuit, 2\diamondsuit, \ldots, K\diamondsuit,$$
$$A\heartsuit, 2\heartsuit, \ldots, K\heartsuit, A\spadesuit, 2\spadesuit, \ldots, K\spadesuit.$$

Mentally label the three cards L (low), M (medium), and H (high) w.r.t. this ordering.

Fitch Cheney's Five-Card Trick

Order the 6 permutations of L,M,H by rank, i.e., 1 = LMH, 2 = LHM, 3 = MLH, 4 = MHL, 5 = HLM and 6 = HML.

Finally, order the three cards in the pile from left to right according to this scheme to communicate the integer desired.

For example, if you were playing the J♣ and trying to communicate the 2♣ to your confederate, then $k = 4$, and you would play the other three cards in the order MHL.

Your confederate knows that the hidden card is a Club, decodes the MHL as 4, and mentally counts 4 past the visible J♣ (mod 13) to get the 2♣.

Fitch Cheney's Five-Card Trick

A weakness in the method as described above is the invariant use of the first position in the pile as the "suit giver," this is soon spotted by alert audiences if the trick is repeated.

Here is a better idea:

Since everybody can view the four cards in the pile, first sum their values and reduce mod 4 (using 4 if you get 0). Now use that number for the position in the display of the suit determining card. Your confederate can also sum mod 4 and figure out which card is special and which three tell her how far to count up from that value.

E.g., a Jack, 8, 2 and 7 gives $11 + 8 + 2 + 7 = 0 \pmod 4$, so the fourth slot would be used for the suit determining card.

Fitch Cheney's Five-Card Trick

Suppose the cards you are handed are 2♣, 2♡ 8♠, 7♢, and J♣.

You play the J♣, in the 4th slot, and communicate $k = 4$ (hence the necessity of counting around to 2♣) using the other three cards as follows:

In standard LMH order they are 7♢, 2♡, 8♠, so in MLH order they are 2♡, 7♢, 8♠,

So lay out the cards in this order: 2♡, 7♢, 8♠, J♣.

Fitch Cheney's Five-Card Trick

Martin Gardner mentions this trick briefly in his 1956 book *Mathematics Magic and Mystery* (Dover), and also in the Scientific American "Mathematical Games" column collection *The Unexpected Hanging and Other Mathematical Diversions*, citing W. Wallace Lee's book *Math Miracles* (1950).

There, it is featured as "Telephone Stud" and attributed to William Fitch Cheney, Jnr, Chairman, Department of Mathematics, University of Hartford, Hartford, CT.

Thanks to Art Benjamin for providing this source, and Paul Zorn for alerting us to the existence of the trick in the first place.

Fitch Four Glory

A volunteer from the crowd chooses any four cards at random and hands them to you. You glance at them briefly, and hand one back, which the volunteer then places face down to one side.

You quickly place the remaining three cards in a row on the table, some face up, some face down, from left to right.

Your confederate, who has not been privy to any of the proceedings so far, arrives on the scene, looks at the cards on display, and promptly names the hidden fourth card—even in the case where all three cards are face down!

How can the pigeonhole principle help us this time?

Fitch Four Glory

Redefine the pigeonholes first!

Partition the stardard deck into three new suits of 17 cards each, leaving one special card aside.

Each of these new suits consists of the one of the standard suits ♣, ♢, ♡ supplemented with four ♠'s.

Specifically,

Suit Alpha is A♣, 2♣, . . . , K♣, 2♠, 3♠, 4♠, 5♠

Suit Beta is A♢, 2♢, . . . , K♢, 6♠, 7♠, 8♠, 9♠

Suit Gamma is A♡, 2♡, . . . , K♡, 10♠, J♠, Q♠, K♠

Fitch Four Glory

Note that A♠ has been left out in the cold: if this special card is among those handed to you, simply hand it back—after a suitable pause—and play the other three face down!

Otherwise, the pigeonhole principle guarantees that (at least) two of the four cards are from one of the three redefined suits, without loss of generality Suit Alpha.

Hand one back, and by placing the remaining three cards on the table in some particular fashion you'll reveal to your confederate the identity of the hidden card.

As before, the basic strategy is to save the "lower" card from Suit Alpha, and communicate the "higher" one, whose numerical value is k past the one you hold on to, where this time k is an integer between 1 and 8 inclusive.

Fitch Four Glory

In the convention we explain below, at least one card will be face up, so once more we can use a face up card (the first such if there are two) to communicate the suit.

The placements UDD, DUD, DDU (one U in 1st, 2nd or 3rd position) and DUU, UDU, UUD (one D in 1st, 2nd or 3rd position), respectively, can be used to tell your confederate that is k is $1, 2, 3, 4, 5$ or 6.

This time we also need a way to communicate 7 or 8 ...

Note: we also have the UUU option at our disposal!

Fitch Four Glory

If we agree to use one particular U (say, the middle one) to give the suit, there are two ways to play the other two: Low-High (to convey $k = 7$) or High-Low (for $k = 8$) w.r.t. some total ordering of the deck, such as lining up Suits Alpha, Beta, Gamma in that order.

Michael Trick at Carnegie Mellon kindly put together a website illustrating this, er, trick in action, it's at `http://mat.gsia.cmu.edu/CARD/`.

Generalizations in a different direction

A popular generalization of the Fitch Cheney result, sticking with the original idea of displaying all cards face up, has been reinvented several times over the past decade or two, by Elwyn Berlekamp and others.

> *For a deck of $n! + n - 1$ cards, if we chose any n of them at random, it is possible to arrange some $n - 1$ of them in a row in such a way as to indicate the identity of the remaining one.*

This can be proved as an application of Hall's Marriage Theorem!

The Fitch Cheney trick considered above is the case $n = 5$. It follows that this can in fact be done with a deck of size 124.

Or with a regular deck plus a toss of a coin . . .

Hamming code applications: Inspiration

Based on an (autumn 2002) idea of colleague Jeffrey Ehme, who may well have been inspired by creations of

Alex Elmsley

(2 March 1929 - 8 January 2006)

British magician/computer programmer. Studied physics and maths at Cambridge; he was secretary of the Pentacle Club there.

Elmsley's groundbreaking work on perfect/faro shuffles (and their fascinating tie in with binary arithmetic) goes back to the 1950s, and forms the basis for some of the finest ever examples of mathematical magic.

Hamming It Up (with a deck of cards)

Aodh takes out a deck of cards, shuffles it, and then splits the deck into two halves, holding one half in each hand, face down.

He invites a spectator to pick out any card, from either pile, and place it face down on the table. This is repeated three times, until there are four cards in a row on the table.

Next, Aodh himself selects four more cards and place them beside the first four, chosing "at random" between the two piles. There is now a row of eight cards on the table.

Aodh stresses that nobody could possibly know what any of them are! (This is true . . .), and turns away.

Hamming It Up

Aodh has the spectator to hand him any of the cards on the table, which he then replaces in the deck.

Next, Aodh invites the spectator to choose a different card from the deck, and has its face noted and shown to all (except Aodh).

Finally, Aodh has this new card replaced in the gap in the row, face down, so that once again there are eight cards on the table.

Aodh now turns back, and has the cards flipped over so that (for the first time) all faces are visible.

Hamming It Up

Aodh recalls the fairness of the procedure as he shuffles the rest of the deck: cards were randomly picked—four by the spectator, four by Aodh—without anybody knowing what any of the cards were.

The spectator then decided which card would be set aside, and which one would replace it.

Aodh did not even witness the switch, and has no idea which card on the table was shown around to the audience.

Aodh says, "Let's see if *somebody who has not seen or heard any of what has happened* can do a little magic!"

Bea enters the room for the first time, and after a moment's reflection, correctly picks out the replacement card from the row of eight.

Hamming It Up With a Deck of Cards

Consider the linear Hamming code where the 4-bit binary message (a, b, c, d) is encoded as the 7-bit codeword

$$(a, b, c, d,\ a+b+c,\ a+c+d,\ b+c+d).$$

(The three new bits are sometimes called parity check digits.)

There are $2^4 = 16$ possible messages, coded as follows:

Hamming It Up

message	codeword
0000	0000000
0001	0001011
0010	0010111
0011	0011100
0100	0100101
0101	0101110
0110	0110010
0111	0111001
1000	1000110
1001	1001101
1010	1010001
1011	1011010
1100	1100011
1101	1101000
1110	1110100
1111	1111111

Hamming It Up

The table can be found via a 4×7 generating matrix G.

If these messages are transmitted, and possible errors introduced, then any of $2^7 = 128$ messages could be received, of which 16 are uncorrupted.

The set of 128 possible received messages forms a 7-dimensional vector space over the field $\{0, 1\}$, equipped with the usual metric: the distance between two vectors is simply the number of places in which they disagree.

For each vector, the number of 1's present (i.e., its distance from 0000000) is called its *weight*. The minimum weight of the nonzero uncorrupted codewords is 3, which is also the distance between any two of them.

Hamming It Up

Note: $3 = 2(1) + 1$!

Such a (linear) code can detect and correct single errors.

Let's assume we know for sure that at most one error has occurred.

Our goal is to *decode*, i.e., from the received message R determine the original M.

The usual approach is to compute syndromes. Namely, find $S = HR$, where H (a 3×7 matrix) is the "dual" matrix of the matrix G referred to above.

S gives (the binary representation of) the one position in which R and M differ.

Decoding can be much easier, as we will soon see.

There are two cases ...

The number of 0s/1s is odd
(if and only if $a + b + c + d = 1 \text{ mod } 2$).

5th bit is 1 iff $d = 0$,

6th bit is 1 iff $b = 0$,

7th bit is 1 iff $a = 0$.

The number of 0s/1s is even
(if and only if $a + b + c + d = 0 \text{ mod } 2$).

5th bit is 1 iff $d = 1$,

6th bit is 1 iff $b = 1$,

7th bit is 1 iff $a = 1$.

Improved Coding Algorithm

In summary

> *"Match the 4th, 2nd, and 1st bits—in that order—iff there are an even number of 0s/1s"*

We now work towards an easy-to-process (in one's head!) decoding algorithm.

Compare the received message R to the 7-bit P ("pretending R is correct") obtained by coding the first four bits of R.

There will be 1, 2 or 3 differences (i.e., perceived errors), among the 5th, 6th and 7th bits only.

For instance, if $R = 1011001$ then $P = 1011010$, a distance of 1 apart, and the perceived error is in the 6th bit.

If the 1st bit of R is actually wrong (i.e., different from that of M), then the 5th and 6th bits of P are affected, so that R and P differ in precisely the bits which are determined by the 4th and 2nd bits of R.

In this case, R and P are 2 apart, and agree in the 7th bit, which is the one determined by the 1st bit of R.

Similarly, if the 2nd bit of R is wrong, R and P are 2 apart, and agree in the 6th bit, which is the one determined by the 2nd bit of R; and if the 4th bit of R is wrong, R and P are 2 apart, and agree in the 5th bit, which is the one determined by the 4th bit of R.

On the other hand, if the 3rd bit of R is wrong, then the 5th, 6th and 7th bits of P are all affected, so R and P are 3 apart.

Finally, if any one of the 5th, 6th, or 7th bits of R are wrong, this is very easy to spot, as the same bit is correct in P, and R and P are 1 apart.

Considering all of the above, one sees that the converses of all of the "if" statements also hold. In summary we have the

Decoding algorithm

“If there appear to be 3 errors in the check digits—comparing ”R” and ”P”—then the actual error in R is in the 3rd bit.

If there appear to be 2 errors, then the actual error in R is in that bit among the 1st, 2nd and 4th bits which corresponds to the whichever one of the 5th, 6th and 7th bits seems okay.

Finally, if there appears to be just 1 error, it is in the position of the actual error in R.”

Hamming It Up—How is this trick done?

First Aodh splits the deck into Red and Black halves, and places these togther, say with the Blacks on top. Some casual in-hand shuffling—of both halves of the deck separately, keeping the blacks on top—will be fairly convincing.

Aodh now separates again, so that the Red cards are in his left hand, the Blacks in his right.

Assume that Black corresponds to 0 and Red to 1. The first four choices, which are made by the spectator, can be viewed as a 4-bit binary message.

The next three choices, which *Aodh* makes, are made in accordance with the binary check bits for the first four choices. Hence, Aodh chooses to use cards from his left or right hand based on the choices made by the spectator.

Hamming It Up

For instance, if the first four cards are three Reds followed by a Black, this corresponds to the message 1110, which is coded as 1110100, and so Aodh selects a Red followed by two Blacks.

The good news is that there is no need for complicated mental calculations; the coding process can be greatly simplified as follows:

The last card which Aodh picks is not related to the Hamming code. It makes no difference what (colour) it is, there is total freedom of choice! (The symmetry of, "You pick four, I pick four" is pleasing, and seems very fair.)

Hamming It Up

Now we move on to the switch: when Aodh turns away and is handed a card to set aside, he peeks at it and note its colour.

It is essential that the replaced card be of the opposite colour—this determines which of part of the deck Aodh offers for that selection!

Recall: Aodh shuffles (properly) before the end of the trick, so that even if the deck is inspected, the evidence is distroyed!

Hamming It Up

There are now two possible scenarios:

(1) **The new card is the eighth.**

This happens if and only if there is *no error* in the 7-bit message corresponding to the colours of the first seven cards—a situation which is very easy to recognise!

(2) **The new card is one of the first seven.**

In this case, it corresponds to a single error in a 7-bit message, which can be detected and corrected by your accomplice as hinted before.

A Better Poker Hand?

Aodh hands a deck to a spectator, and says, “Shuffle the cards well, and deal out some poker hands. Pick one of those hands for yourself, and put the rest aside, we won’t need them. Let me have a look.”

Aodh takes the five cards and moves them around, commenting briefly on the spectator’s luck and prospects.

“I’m going to make you an offer you don’t usually get in real life,” says Aodh, spreading the cards in a neat face-up row. “You may like this hand, you may not. Frankly, it could be better. I’m going to give you the opportunity to replace any one of these cards with a better one! You choose which one you want to get rid of. Take your time, you only get to do this once.”

A Better Poker Hand?

The spectator points to a card, which Aodh slides out. “I promised you a better card did I not? You pick one—you can use any card at all from the rest of the deck, provided it’s the opposite colour of the one you just discarded.” Aodh has the spectator slide the new card into the gap in the row. The discarded card is set then aside.

Aodh call in his accomplice from the next room. “This is Bea, she has studied human psychology a lot. She can figure out which card you switched by asking you to read out the card names in any order you wish and listening to your voice modulations. She says that when you get to the card in question, you’ll get emotional. Be careful, she’s never made a mistake yet.”

Bea delivers!

A Better Poker Hand

The basic idea is to arrange the five cards in a way that they are correctly coded.

> *If we chose any five cards at random from a deck, then it is possible to arrange them in a row in such a way that if one card is exchanged for a new one of a different colour, then the position (and hence identity) of the new card can be detected by somebody who only sees the end result.*

Hamming It Down

Consider the $(2, 3, 5)$ code where the 2-bit binary message (a, b) is encoded as the 5-bit codeword $(a, b, a, b, a + b)$.

There are $2^2 = 4$ possible messages, coded as follows:

message	**codeword**
00	00000
01	01011
10	10101
11	11110

If these messages are transmitted, and possible errors introduced, then any of $2^5 = 32$ messages could be received.

Only 4 of which are uncorrupted.

Hamming It Down

This 5-dimensional vector space over the field $\{0, 1\}$ is equipped with the usual metric, where the distance between two vectors is simply the number of places in which they disagree.

Among the non-zero legitimate codewords, the minimum weight is 3, which is also the distance between any two of them.

This code detects and corrects single errors.

Of the 32 possible received messages, there are 8 with double errors, but we'll focus on the $32 - 4 - 8 = 20$ corrupted messages R arising from the 5 different single-error transmissions for each of the 4 legitimate messages M.

Hamming It Down

These are shown—together with the weights of the received messages R—in first three columns of the next table.

Correct codeword M	Corrupted codeword R	Weight of R	P based on corruption R	Where R, P disagree
00000	10000	1	10101	x, $x+y$
00000	01000	1	01011	y, $x+y$
00000	00100	1	00000	x
00000	00010	1	00000	y
00000	00001	1	00000	$x+y$
01011	11011	4	11110	x, $x+y$
01011	00011	2	00000	y, $x+y$
01011	01111	4	01011	x
01011	01001	2	01011	y
01011	01010	4	01011	$x+y$
10101	00101	2	00000	x, $x+y$
10101	11101	4	11110	y, $x+y$
10101	10001	2	10101	x
10101	10111	4	10101	y
10101	10100	2	10101	$x+y$
11110	01110	3	01011	x, $x+y$
11110	10110	3	10101	y, $x+y$
11110	11010	3	11110	x
11110	11100	3	11110	y
11110	11111	5	11110	$x+y$

Given a received message R, we must determine the original message M.

A study of the table reveals that if the weight of R is 1, then $M = 00000$, whereas if the weight of R is 3 or 5, then $M = 11110$.

While the other half of the cases are not so obvious at first glance, these “weighty” observations are worth bearing in mind when performing associated card tricks.

Let’s take another approach, which turns out to be easier and more logical.

The fourth column of the table gives the “pretend” (consistent) codewords P which are obtained from the first two bits of the erroneous Rs.

In other words, the 3rd, 4th and 5th entries of each P are the parity check digits $[x, y, x + y]$ of the first two bits $[x, y]$ of the corresponding R.

Finally, the last column of the table identifies the exact locations in which P and R differ—namely which of the parity check digits fail to agree.

These are listed in terms of their constituent entries: $[x, y, x + y]$.

Now a clear pattern emerges, leading to the following.

Decoding Algorithm

Given a codeword R, known to contain exactly one error, use its first two bits to generate and append “pretend” parity check digits, thus forming a self-consistent codeword P.

Now see where R and P differ:

If it's in the 3rd and 5th positions, the error in R is in the 1st bit.
If it's in the 4rd and 5th positions, the error in R is in the 2nd bit.
If it's in the 3rd position only, the error in R is in the 3rd bit.
If it's in the 4th position only, the error in R is in the 4th bit.
If it's in the 5th position only, the error in R is in the 5th bit.

The Basic Trick—20% more cards free!

The audience member is invited to select any three cards from the deck, and lay them in a face-up row on the table.

You supplement this row with three more face-up cards of your own choosing.

The audience member replaces any one of the cards in the row with a new card from the deck, with the proviso that the new card must not be the same colour as the one it is substituted for.

Your accomplice now enters the room for the first time, and soon identifies which card in the row was switched.

The Basic Trick

Here's how it works. Your three choices are based on the colours of their first two; their third card colour (denoted by X below) is ignored. Use the convention that 0 represents black, and 1 represents red, and code $[a, b, x]$ as $[a, b, x, a, b, a + b]$. Hence:

If they pick BBX, you pick BBB.
If they pick BRX, you pick BRR.
If they pick RBX, you pick RBR.
If they pick RRX, you pick RRB.

When the switch is made, clearly it's the third card if and only if the others determine a legal codeword (i.e., are self-consistent). Otherwise, one of the 20 cases in the earlier table arises (for the five positions/cards omitting the third one).

The Basic Trick

The decoding can be used to determine which colour sequence was in place before the switch, and hence which card was switched.

For example, suppose the audience member picks 10♣, Q♢, 2♡, and you—concerned only with the colours of their first two selections—pick J♠, 5♡, 7♢.

Now let's assume the spectator discards the second card, Q♢, and in its place puts 8♣.

Mentally, Bea ignores the third card for now, and converts the other five colours to $R = 00011$. She notices that this is not self-consistent, so the ignored third card is not the problem. She has also just worked out $P = 00000$, which differs from R in the b and $a + b$ positions. Hence, she knows that the error in R is in the second bit, and so the 8♣ must be the switched card.

The “given any five cards” scenario:

For the four possible codings (of 2 given cards coded by 3 extra ones)

message	codeword
00	00000
01	01011
10	10101
11	11110

we get (assuming 0 = black and 1 = red) rows of five cards, with no reds, three reds or four reds.

So, should Aodh be handed five cards with one of those possible numbers of reds, Aodh is home and dry. (He even has two ways to arrange the cards in the three reds case.)

But what if Aodh receives one red, two reds or five reds?

The “given any five cards” scenario:

Then he has four, three or zero black cards, since of course $5 - \{1, 2, 5\} = \{4, 3, 0\}$.

He must change the convention: now $0 =$ red and $1 =$ black!

Fortunately this can be done in such a way that Bea *knows* about the convention switch.

There are six cases to consider, three good and three not-so-bad: