> **"Your accomplice must be able to learn *something* when she surveys the scene, if she is to be able to identify three unseen cards! What's surprising is how little information you have to pass to her."**

An ESPeriment with Cards

Colm Mulcahy
Spelman College

It's been roughly seventy years since J.B. Rhine claimed that Extra Sensory Perception (ESP) abilities were exhibited by certain subjects who predicted—significantly more often than by mere chance—the next card in a random arrangement of twenty-five so-called ESP cards. Consider the following demonstration of ESP:

Casually shuffle a packet of cards overhand while talking about an ESP experiment you are about to perform. Then you deal cards into a pile on the table until somebody calls stop, whereupon you riffle shuffle that pile into the remaining cards. The packet is fanned to demonstrate how mixed up it is, and the audience is given a choice of using either the top five cards or the bottom five cards. These five cards are placed in a face-down row from left to right, without altering their order. The rest of the packet of cards is pushed to one side.

"It would be an absolute miracle if anybody outside this room could identify all five of those cards correctly," you proclaim. "Let's try something a little less ambitious, though still impressive in its own way." Picking up two of the cards, you replace them in the packet and shuffle further. You leave the packet face-down on the table—having first closed up any remaining gaps in the row of three face-down cards, without disturbing their order—and then turn away.

A person who claims to have special ESP talents now enters the room for the first time, having seen nothing so far, and surveys the scene. Acting on her trusty ESP instincts, she concentrates hard, and soon identifies all three face-down cards correctly.

A trick along these lines can be performed with ordinary playing cards, but it seems appropriate to use so-called ESP cards here.

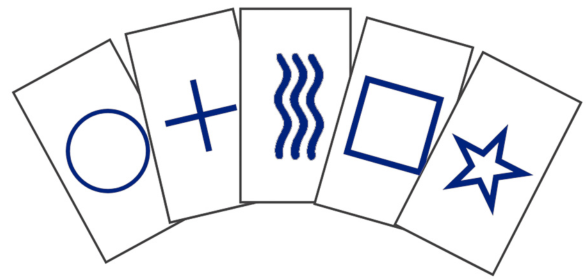

Figure 1. The ESP cards.

There are five distinct ESP cards: circle, plus sign, waves, square, and star. We list them in this order for a simple reason: to establish a one-to-one correspondence with the numbers 1, 2, 3, 4 and 5. Think of the circle as having one bounding curve, the plus sign as two crossed lines, the three wavy lines, and so on.

This trick is largely mathematical, with three distinct components. As we explain it below, there is some nonmathematical signalling going on in one of the three components. The so-called ESP expert is of course your accomplice, with whom you have established communication conventions ahead of time.

Clearly, your accomplice must be able to learn *something* when she surveys the scene, if she is to be able to identify three unseen cards! What's surprising is how little information you have to pass to her. We also suggest an alternative presentation which is 100% mathematical, and can be performed in email or over the telephone.

At first it appears that the three face-down cards could be in any of $5^3 = 125$ possible configurations. Actually there are only 10 possibilities, and you subtly signal the correct one to your accomplice, by means of the "casual" placement of the face-down packet (more on that later in Stage Three).

Proving that this trick can indeed be done—not to mention mastering its actual performance—involves three distinct lemmas. Individually, each step is simple enough, it's the combined result that is far from obvious.

Stage One: Initially, stack the cards in cyclic order, 1234512345123451234512345. The "casual overhand" shuffling performed at the outset must consist of moving clumps of card *en masse* from one end of the packet to the other, which is rather like spinning a wheel: it merely cycles the packet around and whenever you stop it still consists of five sets of five cards, each in the same order, stacked together, no matter what card ends up on top.

The dealing of some cards to the table, reversing their order, and then riffle-shuffling these into the rest of the cards, is an example of a discovery from the mid 1960s known as Gilbreath's (Second) Principle: amazingly it results in a packet of twenty-five cards with the property that each group of five cards, starting with the 1st, 6th, 11th, 16th or 21st card, consists of exactly one each of the five basic ESP cards, though in no predictable order. Thus, for instance, the 10th and 11th cards could both be squares, but the 6th to 10th cards, and the 11th to 15th cards, each have no repeats in them. In particular, the top five cards, and the bottom five cards, are guaranteed to have no repeats.

Here is the general version of the Gilbreath principle (for any types of cards):

> *Consider a "packlet" of* s *cards in some fixed order, repeated* t *times to yield a packet of* st *cards. Deal some—typically about half, but it does not matter how many—of these cards into a pile, thus reversing their order, and riffle-shuffle these into the rest of the packet. The resulting packet retains some sense of predictability, in the sense that peeling off* s *cards at a time, starting at the top (or bottom), will always yield some rearrangement of the original "packlet," with no repeats.*

We only need the case $s = t = 5$. (A simpler form, sometimes referred to as the First Gilbreath Principle, is the special case $s = 2$. This has been explored a lot by both magicians and mathematicians, e.g., using a deck of alternating red and black cards, since its discovery in the late 1950s.) But why does it work, in general? The argument below was inspired by magician Stefan Bartelski's presentation at a Georgia Magic Club (I.B.M. Ring 9) event a few years ago.

Consider the top s cards after the riffle-shuffle. Some, let's say exactly u of them, fell from the right hand, and hence the other $s - u$ fell from the left hand. While we can't be sure exactly how they are now interspersed, we can say that before the dealing was done, these $u + (s - u) = s$ cards were together, and hence formed a complete set with no repetitions. The same holds for each each of the t groups of s cards down to the bottom s.

The upshot of all of this is that, even after all the shuffling of the twenty-five card packet of ESP cards, the top (and bottom) five cards each consist of exactly one card of each type of card. This is unlikely to be suspected by audiences in general, unless they are already familiar with the Gilbreath Principle!

Stage Two: In Stage One we saw that the five cards selected and laid in a face-down row must consist of one each of the five distinct ESP cards, in some order. Your job is to note the exact order before the cards are laid out; actually you really need to know which two cards you must take away, in order that your accomplice can identify the remaining three in order.

When fanning the card faces to demonstrate that they are indeed well mixed, you must do two things simultaneously: scan the top and bottom five cards quickly, looking for something we will shortly explain, and at the same time casually draw attention to the jumbled state of the intermediate cards. Hopefully there will be two like cards adjacent to each other somewhere that you can point to, in order to convey the (false!) impression that "anything is possible."

The top five cards are distinct, but can be in any order; the same applies to the bottom five cards. Remember the identification with 1, 2, 3, 4 and 5 which we suggested earlier. Imagine that the corresponding numbers are laid out like terms in a sequence, from left to right, in the order given. The following result is what we need:

> *In any arrangement of five distinct numbers, either a rising subsequence of length three exists, or a falling subsequence of length three exists.*

Here is an elementary proof of this. Suppose the five numbers are A B C D E. Without loss of generality we may assume that A < B. We claim that if there isn't a rising subsequence (from left to right) of length three, among these five numbers, then there is a falling one instead. Assume there is no such rising subsequence. We must have each of C, D, and E less than B, as otherwise we'd have a rising subsequence of length three starting with A B. If it happens that C > D, then B C D is a falling subsequence of length three, whereas if D > E, then B D E is a falling subsequence of length three; in either case we are done. One of those subcases must in fact occur, for otherwise we'd have both C < D and D < E, and hence a rising subsequence C D E of length three, contrary to our assumption.

This is actually the first non-trivial case of a result which dates back to the 1930s, and is due to the celebrated Paul Erdös and fellow Hungarian Gyorgy Szekeres. It says, more generally, that

> *Any list of $k^2 + 1$ distinct numbers contains an increasing subsequence of size $k + 1$ or a decreasing subsequence of size $k + 1$.*

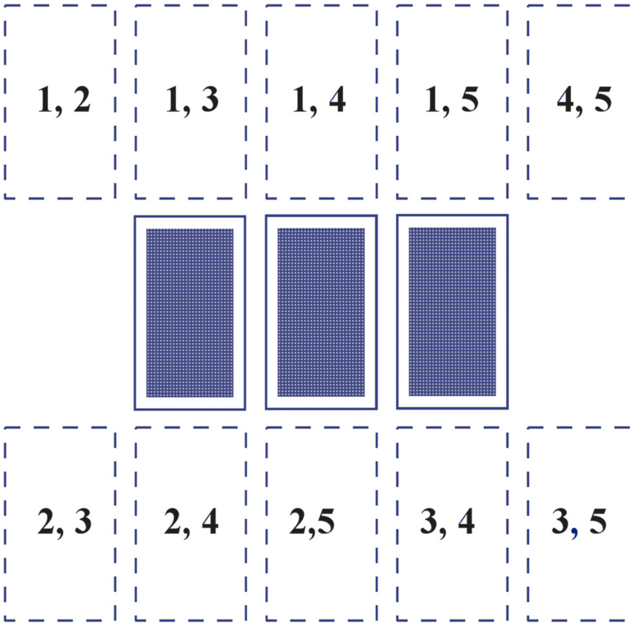

Figure 2. 10 possible place markers for the packet, and a row of three face-down cards.

This is not hard to prove using the pigeonhole principle. (The basic idea is to assign each number an ordered pair (i, j) where i and j are, respectively, the length of the longest increasing subsequence starting from that number and the length of the longest decreasing subsequence starting from that number. Argue that any two (i, j) pairs must be different, and hence by the pigeonhole principle there must be an ordered pair with a coordinate greater than k.) We only need the special case $k = 3$ and $n = 5$ proved above.

How does this help us? Here's the basic idea: no matter which (top or bottom) packet of five cards is used, by quickly scanning the card faces before you deal, you can pick out a rising (or falling) subsequence of length three. By either dealing the packet of five cards from the top, which looks natural but actually reverses their order, or fanning them out face-down, which preserves their order, you can be sure to end up with an arrangement that has three rising cards in it. Either way, you boldly assert, "I am laying these out without changing their order."

The cards you remove, and shuffle into the rest, are of course the two which are *not* in the rising subsequence. This leaves three cards which rise from left to right, and your accomplice will be able to name them correctly, and in order, as soon as she knows their identity as a set.

Stage Three: There are ten ways to select three cards from five cards. Establishing a one-to-one correspondence between 10 ways to signal subtly to your accomplice, and the 10 possibilities for the two missing cards, enables you to communicate to her which three cards remain face-down. Since she knows what order they must be in, the trick can then be concluded successfully.

You signal your accomplice by placing the deck of unused cards in one of ten possible locations near the face down cards, as shown in Figure 2.

For instance, to indicate that the missing cards are the waves (3) and square (4), you would casually place the packet below the third card. To indicate that the missing cards are the circle (1) and plus sign (2), you would casually place the packet above and to the left of the first card.

For a totally mathematical trick, instead of positioning the packet in a sneaky way, simply show your accomplice the two retained cards.

Another approach is to place the two retained cards at the bottom of the remaining twenty cards and shuffle wildly, while retaining those two in place. The packet can then be given to your accomplice, who holds them by her side casually, but finds a moment to absentmindedly peek at the bottom two cards while awaiting for the ESP to kick in!

Needless to say, other possibilities may present themselves to readers who are in possession of exactly ten fingers. ■

Further Reading

For more mathematical card tricks, including more based on Erdös-Szekeres result and the Gilbreath Principles, see the author's Card Colm at `maa.org`.