Low Down Triple Dealing

By Colm Mulcahy

Dedicated to Martin Gardner on the occasion of his 90th birthday

Consider the following three demonstrations of mathemagic:

1. A deck of cards is handed to a spectator, who is invited to shuffle freely. She is asked to call out her favourite ice-cream flavour; let's suppose she says, "Chocolate." Next, she is asked to cut off about a quarter of the deck and hold it ready for dealing. You take another quarter of the deck and demonstrate a spelling deal, dealing cards into a pile, one for each letter in the word "chocolate," before dropping the rest of your quarter deck on top. Set those cards aside and have the spectator perform this spelling routine three times with the cards in her hands. You correctly name the top card in her pile at the conclusion of her triple dealing.

2. A deck of cards is handed to two spectators, each of whom is invited to shuffle at will and then choose a card (of not too low a value) and place it face up on the table. Let's suppose that 4♣ and 9❤ are selected and displayed. You run through the deck face up, tossing out all of the Aces, 2s and 3s — saying, "Sorry, I should have eliminated the low cards earlier." Then riffle shuffle a few times. Remark, "Since a 9 was selected, let's count out nine cards," dealing into a pile on the table. Shuffle overhand and continue, "We'll need four more," as you peel off that many cards as a single unit, without changing their order. Drop these on top of the other nine. (The rest of the deck is ignored from now on.) Pick up this pile of thirteen cards and demonstrate a deal of the nine top cards into a pile, reversing their order, and then putting the remaining four on top. Have the first spectator do this deal three more times, and hand the cards to the second spectator. Have the second volunteer deal either four or nine cards into a pile, with the remainder placed beside this to form a second pile. Recap: the two numbers (4 and 9) being used were determined by freely selected cards, and as a result a deal of nine cards was performed done (three times) on a packet of thirteen cards, which was then split into two piles. Draw attention to the two cards originally selected. Say, "Wouldn't it be surprising if, after all that triple dealing based on the values of two randomly selected cards from a shuffled deck, there were cards intimately related to the two you selected at the bottoms of the two piles now on the table?" Have the piles on the table turned over: one of the cards exposed is 9♣ and the other is 4❤. "A curious alignment with the selected cards."

3. Have each of three volunteers in turn pick a card at random, and then have the cards returned to anywhere in the deck. Shuffle with abandon. Ask a fourth person to name their favourite magician. Assume they say, "Harry Houdini." Hold the deck in the right hand, and peel cards off the bottom into a pile in the left hand, without altering their order, one for each letter, as you spell out the whole name. Hand the stack of twelve cards to the first volunteer and ask him to spell out HOUDINI while dealing out seven cards, then dropping the other five on top. Now give the cards to the second volunteer and give the same directions, and finally to the third volunteer for one last deal of the same type. Take the cards behind your back and immediately produce three cards, handing one to each volunteer face down. Have the chosen cards named, as they are turned over, to reveal that you have correctly located each one.

The same purely mathematical principle underlies each of these demonstrations, with a little more magic thrown in for good effect as we progress to the second and third tricks. We gradually reveal this principle below, and discuss how each of the tricks is done as we go, before finally explaining why the principle works.

Let's start with the first effect. There are two secrets working behind the scenes for you here: an unadvertised but important relationship between the length of the word being spelled out and the size of the "quarter deck" which the spectator starts with, and the fact that you must somehow know the identity of one card in the spectator's hand from the beginning! It should come as no surprise that the card in question is the bottom card: asking the spectator to hold the cards in her hand in preparation for the spelling is just to give you an added opportunity to peek at this card, if you haven't already done that as she completed her shuffling. You must do whatever it takes to discover that card's identity!

This is the scoop on the ice cream trick:

Claim 1: Start with n cards, the bottom one of which is known. If k cards are dealt out into a pile, thus reversing their order, and the remaining $n - k$ cards are dropped on top as a unit, and this type of deal is repeated twice more, then the known card rises like cream to the top — provided that $n \le 2k$.

In the case of the nine-letter word CHOCOLATE, the trick works provided that the portion of the deck selected by the volunteer contains at most eighteen cards. If MINT CHOCOLATE CHIP (seventeen letters) is named, you'll ask for between a third and half of the deck. (If RUM is selected, try to force RUM RAISIN!)

The triple deal described is actually 75% of a rather interesting quadruple deal. This is the real scoop:

Claim 2: Start with n cards, and assume that $n \le 2k \le 2n$. If k cards are dealt out into a pile, thus reversing their order, and the remaining $n - k$ cards are dropped on top as a unit, and this deal is *repeated three more times*, the entire packet of n cards is restored to its original order.

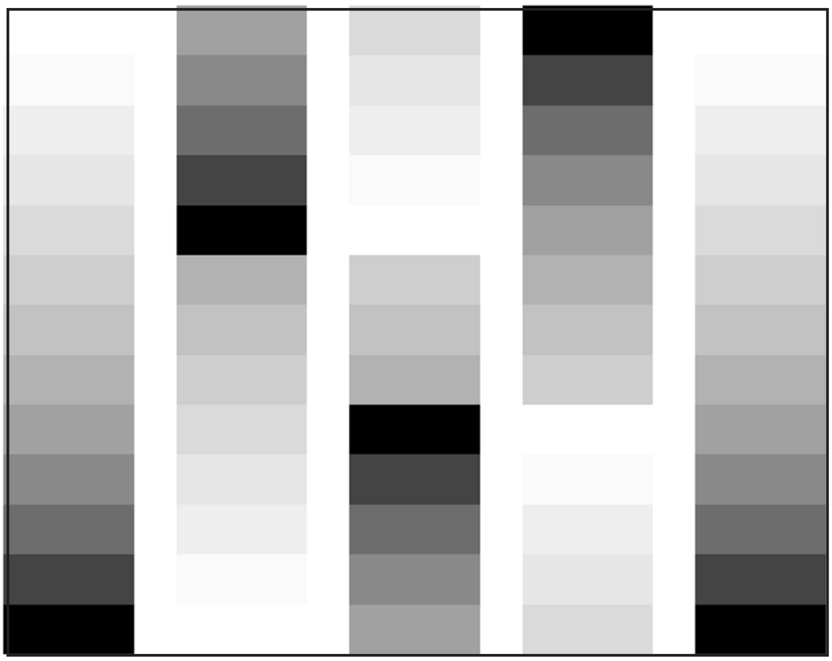
Now consider the second effect above. Two cards (not Aces, 2s or 3s) are chosen and set aside face up. Let's suppose they are 4♣ and 9❤. As you run through the deck face up, ostensibly to toss out the low valued cards, what you really focus on doing is cutting the 9♣ and 4❤ to the top and bottom respectively. They will stay there if you are careful how you riffle shuffle. Continue as described earlier: reversing nine cards into a pile and then doing some overhand shuffling whose purpose is to bring the bottom card to the top. Peel four more off the top without reversing them and drop on top of the other nine. You now have thirteen cards with the desired two cards at the top and bottom of that packet. Your subsequent demonstration of dealing nine and dropping four is just the first of a series of four deals: the first spectator does the next three deals, thereby restoring the packet to its initial state. The second spectator deals (either four or nine) cards into a pile and then there are two piles on the table with one of the desired cards at the bottom of each pile. You are all set for the grand finale.

The third effect uses the fact that after three deals of the type described, not only does the bottom card rise to the top, the next to last card becomes the second card from the top, the one above that becomes the third card, and so on:

This is the real triple scoop:

Claim 3: If k cards from n cards are dealt out into a pile, reversing their order, and the remaining $n - k$ are dropped on top as a unit, and this process is repeated twice more, then provided that $n \geq k \geq n/2$, the original k bottom cards become the top k cards, in reverse order.

To perform the third trick, ask each of three volunteers to pick a card at random. Have these cards returned, one at a time, to the deck and *then control them to the bottom* — this means that you appear to allow free choice of where to put the cards, but you actually use elementary magic techniques (e.g., double cuts) to get each card to the bottom. As a result, the third volunteer's card is at the bottom of the deck, the second volunteer's card is one up from the bottom, and the first volunteer's card is two up from the bottom. Peel cards off the bottom of the deck — without altering their order — one for each letter of the name of the magician called out, as you spell out both words in full. Hand the resulting packet of cards to the first volunteer and ask that the longer of the two names (HOUDINI in our example) be spelled out as cards are dealt into a pile, before dropping the remainder on top. Now give the cards to the second volunteer and finally to the third volunteer for two more deals. The three chosen cards are now on the top of the packet of cards, with the order reversed, and you are all set to conclude in triumph.



Proof Without Words

Why are all of the above claims valid for any n and k with $n \leq 2k \leq 2n$? It's certainly easy to see if $n = k$ (reversing all of the cards each time), and almost as easy to see if $n = 2k$ (reversing exactly half of the cards). Actually, it's *easy to see* in all cases: Suppose for the sake of concreteness that $n = 13$ and $k = 8$. Let's agree to represent a pile of thirteen cards in a particular order by a sequence of grayscale panels in increasing order of brightness, from black for the top card to white for the bottom card, as depicted in the image.

Then the results of the four deals — each of eight cards into a pile with the other five dropped on top — is given by the successive images in the pictures.

Since the last image shows a fully restored pile, the deal in question has period 4: after four deals we are always back to where we started. After three such deals, the original bottom card (white) has risen to the top — in preparation for its final journey back to the bottom under one more deal. Moreover, it is clear that the eight bottom cards become the eight top cards, suitably reversed, after three deals. There are just three portions of the packet — of sizes 5, 3 and 5 here — to keep track of: and they move around intact, subject at most to some internal reversals. The only relationship between 13 and 8 which is needed to make this sequence of images totally generalizable is the fact that $8 \geq 13/2$.

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